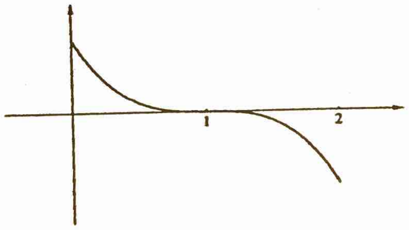
1996

#1

Solution	Scoring Scale Points
(a) $x = -2$ $f'(x)$ changes from positive to negative at $x = -2$, or f is increasing to the left of $x = -2$ and decreasing to the right of $x = -2$.	$2 \begin{cases} 1: x = -2 \\ 1: \text{reason} \end{cases}$
(b) $x = 4$ $f'(x)$ changes from negative to positive at $x = 4$, or f is decreasing to the left of $x = 4$ and increasing to the right of $x = 4$.	$2 \begin{cases} 1: x = 4 \\ 1: \text{reason} \end{cases}$
(c) $(-1, 1)$ and $(3, 5)$ f' is increasing on these intervals.	$3 \begin{cases} 1: (-1, 1) \\ 1: (3, 5) \\ 1: \text{reason using } f' \end{cases}$ max 1/3 for one incorrect interval 0/3 for two incorrect intervals
(d) 	$2 \begin{cases} <-1> f(1) \neq 0 \\ <-1> \text{not decreasing} \\ <-1> \text{incorrect concavity} \\ <-1> f'(1) \neq 0 \end{cases}$

Question 2

In part (a), the student is asked to find the area of a region. In part (b), the student is asked to find the equation of the vertical line that divides the region in half. Part (c) asks for the volume of the solid whose base is the given region and whose cross sections are squares. It is expected that the student will use a calculator to graph the region to help visualize the problem. The student is permitted, but not required, to use a calculator to compute the required definite integrals.

2. Let R be the region in the first quadrant under the graph of $y = \dfrac{1}{\sqrt{x}}$ for $4 \le x \le 9$.

(a) Find the area of R.

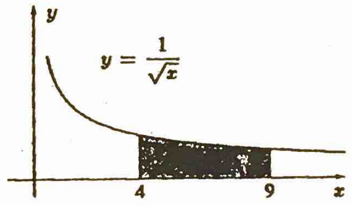
(b) If the line $x = k$ divides the region R into two regions of equal area, what is the value of k?

(c) Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis are squares.

Solution | *Scoring Scale*

Points

(a) $\displaystyle\int_4^9 \frac{dx}{\sqrt{x}} = 2$



3
- 1 : limits
- 1 : integrand
- 1 : answer
  - 0/1 if integrand is incorrect

(b) $\displaystyle\int_4^k \frac{dx}{\sqrt{x}} = 1$

$$2\sqrt{x}\Big|_4^k = 1$$

$$2\sqrt{k} - 2\sqrt{4} = 1$$

$$k = \frac{25}{4}$$

$$\left(\text{or } \int_k^9 \frac{dx}{\sqrt{x}} = 1 \text{ or } \int_4^k \frac{dx}{\sqrt{x}} = \int_k^9 \frac{dx}{\sqrt{x}}\right)$$

3
- 1 : $\displaystyle\int_4^k \frac{dx}{\sqrt{x}}$ or $\displaystyle\int_k^9 \frac{dx}{\sqrt{x}}$
- 1 : equation involving the two halves of R
- 1 : answer
  - 0/1 if answer from equation not involving relevant areas

(c) $\displaystyle\text{volume} = \int_4^9 \left(\frac{1}{\sqrt{x}}\right)^2 dx$

$$= \int_4^9 \frac{dx}{x} = \ln x\Big|_4^9 = \ln\frac{9}{4}$$

(or 0.811)

3
- 1 : limits
- 1 : integrand
- 1 : answer
  - 0/1 if integrand is incorrect

#3

Solution	Points	Scoring Scale
(a) $S(t) = Ce^{kt}$ $S(0) = 6 \implies C = 6$ $S(5) = 12 \implies 12 = 6e^{5k}$ $2 = e^{5k}$ $k = \frac{\ln 2}{5}$ (0.138 or 0.139)	3	1 : $C = 6$ 1 : $12 = 6e^{5k}$ 1 : $k = \frac{\ln 2}{5}$
(b) average rate $= \frac{1}{13-3}\int_3^{13} 6e^{\frac{\ln 2}{5}t}\,dt$ $= \frac{3}{\ln 2}\left[e^{2.6\ln 2} - e^{0.6\ln 2}\right]$ billion gal/yr (19.680 billion gal/yr)	3	1 : uses [3, 13] and divides by $13-3$ 1 : integrand 1 : answer with units 0/1 if not $\frac{1}{b-a}\int_a^b S(t)\,dt$
(c) $\int_5^7 S(t)\,dt$ $\doteq \frac{1}{4}[S(5) + 2S(5.5) + 2S(6) + 2S(6.5) + S(7)]$	1	Trapezoidal rule with S, $n = 4$, interval [5, 7]
(d) This gives the total consumption, in billions of gallons, during the years 1985 and 1986.	2	1 : total consumption in a time period 1 : correct time period liquid measure 0/2 for "rate of consumption"

Board Note
AB-3, BC-3

Part(b)

Student solves $\frac{S(13) - S(3)}{10} = S'(t)$ for t_0.

$S(t_0) = 19.680$ billions gal/yr

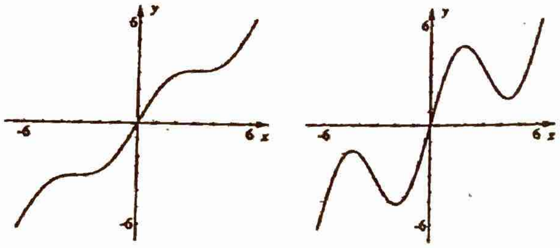
3/3 if correct.

Otherwise, read by standard where max 1 - 0 - 0.

#4

Solution

(a)



(b) $y' = 1 = 1 + b\cos x$

$b\cos x = 0$

$\cos x = 0 \quad \Rightarrow \quad x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

$y = x + b = x + b\sin x$

$b = b\sin x$

$1 = \sin x$

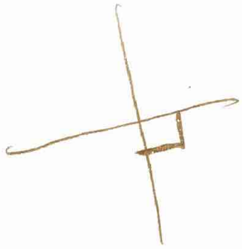
$x = -\dfrac{3\pi}{2}$ or $\dfrac{\pi}{2}$

(c) No, because $f'(x) = 1$ (or $f'(x) \neq 0$) at x-coordinates of points of tangency.

(d) $f''(x) = -b\sin x = 0$

$\sin x = 0$

$f(x) = x + b\cdot 0 = x$
at x-coordinates of any inflection points



Scoring Scale

Points

2
- 1 : graph of $y = x + \sin x$ (increasing, through origin, cd - cu - cd - cu)
- 1 : graph of $y = x + 3\sin x$ (inc - dec - inc - dec - inc, through origin, cd - cu - cd - cu)

4
- 1: derivatives of $x + b$ and $x + b\sin x$
- 1: sets derivatives equal
- 1: sets y-values equal
- 1: answer from merged information

max: 0 - 1 - 1 - 0 if specific value(s) for b

max: 0 - 0 - 1 - 0 if no derivative

1: answer with reason

2
- 1: sets second derivative equal to 0
- 1: shows $f(x) = x$

0/2 if specific value(s) for b

#5

Solution

(a) volume $= V = \pi \int_0^9 \frac{25}{3}\sqrt{y}\,dy = 150\pi \text{ ft}^3$

(or 471.238 ft^3 or 471.239 ft^3)

$$V = 2\pi \int_0^5 x\left(9 - \frac{9}{625}x^4\right) dx$$

(b) time $= \dfrac{\text{volume}}{\text{rate}} = \dfrac{150\pi}{8}$

therefore, 59 minutes

(c) $V = \pi \int_0^h \frac{25}{3}\sqrt{y}\,dy$

$$\frac{dV}{dt} = \frac{25}{3}\pi\sqrt{h}\,\frac{dh}{dt}$$

$$\frac{dV}{dt} = 8$$

when $h = 4$, $8 = \dfrac{25}{3}\pi(2)\dfrac{dh}{dt}$

$$\frac{dh}{dt} = \frac{12}{25\pi} \text{ ft/min}$$

(or 0.152 ft/min or 0.153 ft/min)

Scoring Scale

Points

$$3\begin{cases} 2:\text{integral}\begin{cases} 1:\text{limits and } \pi/2\pi \\ \quad \pi\int_0^9 (\;)\,dy,\quad 2\pi\int_0^5 (\;)\,dx \\ 1:\text{integrand} \end{cases} \\ 1:\text{answer with units} \end{cases}$$

not eligible for answer point unless first two points are earned.

1: integer answer

$$5\begin{cases} 1:\text{volume as definite integral using } h \\ 2:\text{finding } \dfrac{dV}{dt}\begin{cases} 1: \dfrac{dV}{dh} \\ 1: \dfrac{dV}{dt} = \dfrac{dV}{dh}\cdot\dfrac{dh}{dt} \end{cases} \\ 1: \dfrac{dV}{dt} = 8 \\ 1:\text{answer with units} \end{cases}$$

if V is linear, max 2/5 (0 - 0 - 1 - 1 - 0)
if V is constant, max 1/5 (0 - 0 - 0 - 1 - 0)

#6

Solution

(a) let Q be $\left(a, a - \frac{a^2}{500}\right)$.

$$\frac{dy}{dx} = 1 - \frac{x}{250}$$

setting slopes equal:

$$1 - \frac{a}{250} = \frac{\left(a - \frac{a^2}{500}\right) - 20}{a}$$

$$a = 100$$

or

$$\frac{dy}{dx} = 1 - \frac{x}{250}$$

equation for $\ell : y = \left(1 - \frac{a}{250}\right)x + 20$

setting y-values equal:

$$\left(1 - \frac{a}{250}\right)a + 20 = a - \frac{a^2}{500}$$

$$a = 100$$

(b) $y = \frac{3}{5}x + 20$

(c) height of hill at $x = 250$:

$$y = 250 - \frac{250^2}{500} = 125 \text{ feet}$$

height of line at $x = 250$:

$$y = \frac{3}{5}(250) + 20 = 170 \text{ feet}$$

Yes, the spotlight hits the tree since the height of the line is less than the height of the hill + tree which is 175 feet.

Scoring Scale

Points

4 $\begin{cases} \end{cases}$

- 1 : slope of tangent line from parabola
- 1 : uses the condition that $\left(a, a - \frac{a^2}{500}\right)$ is on line ℓ
- 1 : uses the condition that slopes are equal at Q
- 1 : answer
  0/1 if student is solving an irrelevant equation

2

- 1 : slope
  0/1 if $m \le 0$
- 1 : equation

3

- 1 : height of hill
- 1 : height of line
  0/1 if height < 0
- 1 : answer with analysis